Permutads

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Planar rooted trees

Associahedron $\mathcal{A}_n$ is a $n-1$ dimensional polytope, whose faces of dimension $r$ correspond to planar rooted trees with $n+1$ leaves and $n-r$ internal vertices.

Let $T^r_n$ denotes the set of planar rooted trees with $n+1$ leaves and $r$ internal vertices,

$$\begin{align*}
T^0_0 &= \{ \emptyset \} \\
T^1_1 &= \{ \Y \} \\
T^2_2 &= \{ \Y \Y \} \\
T^1_2 &= \{ \Y \Y \} \\
T_2 &= \{ \Y \Y \} \\
T_1 &= \{ \Y \} \\
T_0 &= \{ \} \\
\end{align*}$$
Associahedron or Stasheff polytope

\[ A_2 \]

\[ A_3 \]
Algebraic structures associated to the Stasheff polytope

- Pre-Lie system is a colored operad (i.e. the operations are not always defined, they depend on the degree of the elements). Pre-Lie systems are equivalent to non-$\Sigma$ operads, modulo a shift of the degree. We add a section describing non-$\Sigma$ operads as monads in certain category of functors, following V. Ginzburg and M. Kapranov).

- Tridendriform is a non-$\Sigma$ operad, which extend the notion of dendriform algebras defined by J.-L. Loday.
Pre-Lie systems

Graftings

A pre-Lie system or non-symmetric operad is a graded vector space $L = \bigoplus_{n \geq 0} L_n$ equipped with linear maps

$$\circ_i : L_m \otimes L \to L,$$ for $1 \leq i \leq m,$

satisfying

1. $x \circ_j (y \circ_i z) = (x \circ_j y) \circ_{i+j} z,$ for $0 \leq i \leq |y|$ and $0 \leq j \leq |x|,$

2. $(x \circ_j y) \circ_i z = (x \circ_i z) \circ_{j+|z|} y,$ for $0 \leq i < j.$

Remark: Let $(L, \circ_i)$ be a pre-Lie system, then $L$ with the binary product

$$x \circ y := \sum_{i=0}^{|x|} x \circ_i y,$$

is a pre-Lie algebra, as defined by M. Gerstenhaber.

Gerstenhaber’s example: The space of Hochschild cochains of an associative algebra $A.$


Free pre-Lie systems

Let $t$ and $w$ be planar rooted trees, the element $t \circ_i w$ is the tree obtained by grafting the root of $w$ on the $i$-th. leaf of $t$.

**Free objects** Denote by $\text{Pre-Lie}(V)$ the free pre-Lie system spanned by a vector space $V$. The vector space spanned by all planar binary rooted trees, with the $\circ_i$'s, is the free pre-Lie system spanned by one element $\text{Pre-Lie}(K)$. The vector space spanned by all planar rooted trees, with the operations $\circ_i$, is the free pre-Lie system spanned by the graded vector space $\bigoplus_{n \geq 1} Kc_n$. 
In general, let $X$ be a basis of a vector space $V$. Let $\mathcal{T}_n^X$ be the set of planar rooted trees with $n + 1$ leaves and the vertices colored by the elements of $X$ in such a way that each vertex with $r + 1$ inputs is colored by an element of $X$ of degree $r$.

The free pre-Lie system spanned by $V$ is the space spanned by the set $\bigcup_n \mathcal{T}_n^X$ with the product $\circ_i$ given by the grafting at the $i$-th leaf.
Relation with non-$\Sigma$-operads

Let $L$ be a pre-Lie system, define $\mathcal{P}$ as $\mathcal{P}_{n+1} = L_n$, with the partial operation $\circ_i$ and the trivial action of the symmetric group $\Sigma_{n+1}$, then $\mathcal{P}$ is non-$\Sigma$ operad.
Coalgebra structure on a pre-Lie system

Let $L = \bigoplus_{n \geq 0} L_n$ be a pre-Lie system. A coproduct on $L$ is a linear map $\Delta : L \rightarrow L \otimes L$ such that:

$$\Delta(x \circ_i y) = \sum_{|x(1)| < i} x(1) \otimes (x(2) \circ_{i-|x(1)|} y) + \sum_{|x(1)| = i} (x(1) \circ_i y(1)) \otimes (x(2) \circ_0 y(2)) + \sum_{|x(1)| > i} (x(1) \circ_i y) \otimes x(2).$$
If \((V, \Theta)\) is a graded coassociative coalgebra, then Pre-Lie\((V)\) is equipped with a natural coproduct:

1. \(\Delta_\Theta(c_n, x) := \sum_{i=0}^{n} \sum_{|x(1)|=i} (c_i, x(1)) \otimes (c_{n-i}, x(2))\).
2. \(\Delta_\Theta\) is a coproduct for the pre-Lie system structure of Pre-Lie\((V)\).
Let \((L, \circ_i)\) be a pre-Lie system. The products:

- \(x \circ_0 y\),
- \(x \circ_L y = x \circ_{|x|} y\),

are associative.
**Remark:** Note that the relationships satisfied by $\Delta$ and the $\circ_i$’s imply that:

1. $$\Delta(x \circ y) = \sum x(1) \otimes (x(2) \circ y) + (x(1) \circ y) \otimes x(2) + \sum (x(1) \circ_L y(1)) \otimes (x(2) \circ_0 y(2)).$$

2. $$\Delta(x \circ_0 y) = \sum y(1) \otimes (x \circ_0 y(2)),$$

3. $$\Delta(x \circ_L y) = \sum x(1) \otimes (x(2) \circ_L y) + (x \circ_L y(1)) \otimes y(2).$$

$(L, \circ_0^{op}, \Delta)$ and $(L, \circ_L^{op}, \Delta)$ are unital infinitesimal algebras.
Let $(V, \theta)$ be a conilpotent coassociative coalgebra. Define a boundary map on $\text{Pre-Lie}(V)$ as follows:

1. $\delta(x) := \sum (-1)^{|x(1)|} x(1) \circ_i x(2)$, for $x \in V$ with $\theta(x) = \sum x(1) \otimes x(2)$.

2. $\delta$ is a derivation for all the binary products $\circ_i$.

When $V = \bigoplus \mathbb{K}c_n$ is the space spanned by all corollas with the coproduct given by:

$$\theta(c_n) = \sum_{i=0}^{n} c_i \otimes c_{n-i},$$

we get the associahedra as the bar construction.
Non-$\Sigma$ operads as monads (V. Ginzburg and M. Kapranov)

Let $\mathbb{N}^+\text{Vect}$ be the category of positively graded vector spaces. Objects of $\mathbb{N}^+\text{Vect}$ are families $M = \{M_n\}$ of $\mathbb{K}$-vector spaces. Planar rooted trees define a monad in the category $\mathbb{N}^+\text{Vect}$ as follows:

- For $t \in T_n$,
  \[ M_t := \bigotimes_{v \in \text{Vert}(t)} M_{|v|}, \]
  where $|v|$ is the number of inputs of $v$.

- For $M \in \mathbb{N}^+\text{Vect}$, the graded vector space $\mathbb{P}(M)$ is:
  \[ \mathbb{P}(M) := \bigoplus_{t \in T_n} M_t, \]
  and $\mathbb{P}(M)_1 := \mathbb{K}$. 
The map $\nu(M) : M \rightarrow P(M)$ consist in associating to $\mu \in M_n$ the corolla with $n + 1$ leaves, colored by $\mu$.

$P_{n+1}(M)$ is spanned by planar rooted trees with $n + 1$ leaves and the vertices decorated by elements of $M$.

The grafting of trees defines $\Gamma : P \circ P \rightarrow P$

$$\Gamma(t; w_0, \ldots, w_n) := (((t \circ_0 w_0) \circ_{|w_0|+1} w_1) \ldots) \circ_{|w_0|+\ldots+|w_{n-1}|+1} w_n,$$

which is an associative and unital transformation of functors.
\((P, \Gamma, \iota)\) is a monad in the category \(\mathbb{N}^+\text{Vect}\).

A non-\(\Sigma\) operad is an algebra over this monad. That is, an object
\(M = \{M(n)\}_{n \geq 1}\) in \(\mathbb{N}^+\text{Vect}\) with:

\[\mathcal{P}(M) \longrightarrow M,\]

compatible with \(\Gamma\) and \(\iota\).
Permutohedra

The permutohedron $\Psi_n$ is a $n-1$ dimensional polytope whose faces of dimension $r$ correspond to all surjective maps from $\{1, \ldots, n\}$ to $\{1, \ldots, n - r\}$.

$\text{Surj}_n$ is the set of surjective maps defined on $\{1, \ldots, n\}$.

Note that for $r = 0$, we get the set $\Sigma_n$ of permutations of $n$ elements.
Relation with associahedra

There exist canonical maps $\text{Surj}_n \to T_n$ which are graded and surjective.

Idea:

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (1,1) -- (2,0) -- (0,0);
\draw (1,1) -- (2,2) -- (3,1) -- (2,1);
\draw (2,1) -- (2,2);
\draw (2,1) -- (2,0);
\draw (2,0) -- (2,1);
\draw (0,0) -- (2,0);
\draw (1,1) -- (2,1);
\draw (2,0) -- (2,1);
\draw (2,1) -- (2,2);
\end{tikzpicture}
\end{center}

(1, 3, 3, 4, 4, 1, 2)

A. Tonks: The associahedron may be obtained from the permutohedron by contracting some faces.
A **shuffle algebra** is a graded vector space $A = \bigoplus_{n \geq 1} A_n$, endowed with operations:

$$\bullet : A_m \otimes A_n \longrightarrow A_{n+m}, \text{ for } \gamma \in \text{Sh}(n, m)$$

satisfying:

$$x \bullet \gamma (y \bullet \delta z) = (x \bullet \sigma y) \bullet \lambda z,$$

whenever $\gamma \cdot (\delta \times 1_n) = \lambda \cdot (1_r \times \sigma)$ in $\text{Sh}(n, m, r)$. Since any $k$-shuffle $\sigma \in \text{Sh}(i_1, \ldots, i_k)$ can be written as a composition of 2-shuffles, there exists:

$$\bullet : A_{i_1} \otimes \cdots \otimes A_{i_k} \longrightarrow A_{i_1 + \cdots + i_k}.$$
The vector space spanned by all permutations, with the operations:

$$\alpha \bullet_\gamma \beta := (\beta \times \alpha) \cdot \gamma^{-1},$$

is the free shuffle algebra spanned by one element. The vector space spanned by $\bigcup_{n \geq 1} \text{Surj}_n$ is the free shuffle algebra spanned by the maps $c_n = (1, \ldots, 1) : \{1, \ldots, n\} \rightarrow \{1\}$. 
Coalgebra structure

Let $\sigma \in \Sigma_n$ and $0 \leq i \leq n$, there exist unique $\gamma \in \text{Sh}(i, n - i)$, $\sigma_{(1)}^i \in \Sigma_i$ and $\sigma_{(2)}^i \in \Sigma_{n-i}$ such that

$$\sigma = \gamma \cdot (\sigma_{(1)}^i \times \sigma_{(2)}^i).$$

$$\sigma = (1, 4, 3, 6, 2, 5)$$

$$i = 3$$

$$(1, 3, 4, 2, 5, 6) \cdot ((1, 3, 2) \times (1, 3, 2))$$
A shuffle bialgebra is a shuffle algebra $A$ equipped with a coassociative coproduct $\Delta$ such that:

$$\Delta(x \bullet_\sigma y) = \sum_{r=1}^{n+m-1} \left( \sum_{t=1}^{r} (x^{(1)} \bullet_{\sigma^{(1)}_t} y^{(1)}) \otimes (x^{(2)} \bullet_{\sigma^{(2)}_{n+m-r}} y^{(2)}) \right).$$

**Remark:** A pre-Lie system, equipped with a coproduct is a shuffle bialgebra.
**Bar construction**

Let \((V, \theta)\) be a conilpotent coassociative coalgebra. Define a boundary map on \(\text{Shuff}(V)\) as follows:

1. \(\delta(x) := \sum \text{sgn}(\sigma)(-1)^{|x(1)|}x(1) \bullet_{\sigma} x(2)\), for \(x \in V\) with \(\overline{\theta}(x) = \sum x(1) \otimes x(2)\),

2. \(\delta\) is a derivation for all the binary products \(\bullet_{\sigma}\).

When \(V = \bigoplus \mathbb{K}c_n\) is the space spanned by all corollas with the coproduct given by:

\[
\theta(c_n) = \sum_{i=0}^{n} c_i \otimes c_{n-i},
\]

we get the permutohedra as the bar construction.
**Arity**

Let \( \text{Sur}(n, k) \) denotes the set of a surjective maps from \( \{1, \ldots, n\} \) to \( \{1, \ldots, k\} \).

A **vertex** of \( t \in \text{Sur}(n, k) \) is an element in the image of \( t \).

The **arity** of \( v \in \text{Vert}(t) \) is \( |v| := \# t^{-1}(v) + 1 \).

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
1 = v_1 & 2 = v_2 & 3 = v_3 & 4 = v_4 & & & \\
\end{array}
\]

\[
f = (2, 3, 1, 2, 3, 2, 4)
\]
Substitution

Let $t \in \text{Sur}(n, k)$ and $t_j \in \text{Sur}(i_j, m_j)$, $j = 1, \ldots, k$, be surjective maps such that $i_j = \# t^{-1}(j)$. Let $m := \sum_j m_j$. The substitution of $\{t_j\}$ in $t$ is the surjective map $(t; t_1, \ldots, t_k) \in \text{Sur}(n, m)$ given by

$$(t; t_1, \ldots, t_k)(a) := m_1 + \cdots + m_{j-1} + t_j(b),$$

whenever $t(a) = j$ and $a$ is the $b$-th element in $t^{-1}(j)$.
Substitution is associative.
**Monad**

(joint paper with J.-L. Loday, to appear in J. of Combinatorial Theory, Series A)

Surjective maps define a monad in the category $\mathbb{N}^+\text{Vect}$ as follows:

- For $t \in \text{Sur}(n, k)$,

$$M_t := \bigotimes_{v \in \text{Vert}(t)} M_{|v|},$$

where $|v|$ is the number of inputs of $v$.

- For $M \in \mathbb{N}^+\text{Vect}$, the graded vector space $\mathbb{P}(M)$ is:

$$\mathbb{P}(M)_{n+1} := \bigoplus_{t \in \text{Surj}_n} M_t,$$

and $\mathbb{P}(M)_1 := \mathbb{K}$. 

Result: The substitution of surjective maps defines a transformation of functors $\Gamma : \mathcal{P} \circ \mathcal{P} \rightarrow \mathcal{P}$ which is associative and unital. So $(\mathcal{P}, \Gamma, \iota)$ is a monad on graded vector spaces. A permutad is a unital algebra over the monad $(\mathcal{P}, \Gamma, \iota)$. 

Permutad
Applications

1. Study of combinatorial Hopf algebras (Hivert-Novelli-Thibon, Aguiar-Sottile, Lam-Pylyavskyy, ...).
2. Generalized associahedra (Carr, Devadoss, Forcey)
3. Shuffle operads (Dotsenko, Koroshkin)