The Hopf monoid of generalized permutahedra

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Joyals notion of species constitutes a good framework for the study of certain algebraic structures associated to combinatorial objects. We discuss the notion of Hopf monoid in the category of species and illustrate it with several examples. We introduce the Hopf monoid of generalized permutahedra (the latter are certain polytopes recently studied by Postnikov, Reiner and Williams). Our main result is an explicit antipode formula for this Hopf monoid. We explain how reciprocity theorems of Stanley on graphs and of Billera, Jia and Reiner on matroids can be deduced from this result. The talk is based on joint works with Swapneel Mahajan and with Federico Ardila.


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Species

A species P consists of:

- For each finite set I, a vector space $\mathrm{P}[I]$.
- For each bijection $\sigma : I \to J$, a linear map

$$\mathrm{P}[\sigma] : \mathrm{P}[I] \to \mathrm{P}[J]$$

  such that

$$\mathrm{P}[\sigma \circ \tau] = \mathrm{P}[\sigma] \circ \mathrm{P}[\tau] \quad \text{and} \quad \mathrm{P}[\mathrm{id}] = \mathrm{id}.$$

The species of linear orders

Define a species L by:
$\mathrm{L}[I] :=$ vector space with basis the set of all linear orders on I.

$$\mathrm{L}[a,b,c] = \Bbbk\{abc,\ bac,\ acb,\ bca,\ cab,\ cba\}.$$

If $\sigma : \{a,b,c\} \to \{x,y,z\}$ is

$$\begin{array}{ccc} a & b & c \\ \downarrow & \downarrow & \downarrow \\ y & z & x \end{array}$$

then $\mathrm{L}[\sigma] : \mathrm{L}[a,b,c] \to \mathrm{L}[x,y,z]$ is

$$\begin{array}{cccccc} abc & bac & acb & bca & cab & cba \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ yzx & zyx & yxz & zxy & xyz & xzy \end{array}$$

Hopf monoids

A (connected) Hopf monoid $(\mathrm{P}, \mu, \Delta)$ consists of:

- A species P.
- For each $I = S \sqcup T$, maps

$$\mathrm{P}[S] \otimes \mathrm{P}[T] \xrightarrow{\mu_{S,T}} \mathrm{P}[I] \quad \text{and} \quad \mathrm{P}[I] \xrightarrow{\Delta_{S,T}} \mathrm{P}[S] \otimes \mathrm{P}[T].$$

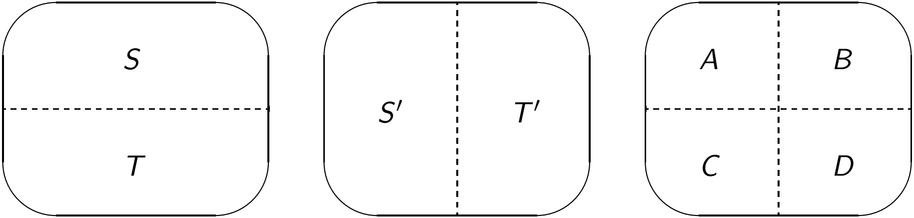
- Two inverse maps

$$\Bbbk \longrightarrow \mathrm{P}[\emptyset] \quad \text{and} \quad \mathrm{P}[\emptyset] \longrightarrow \Bbbk.$$

- Various axioms, including the following:

Compatibility axiom for Hopf monoids

Fix decompositions $S \sqcup T = I = S' \sqcup T'$, and let A, B, C, D be:



Then:

$$\begin{array}{ccc}
P[A] \otimes P[B] \otimes P[C] \otimes P[D] & \xrightarrow{\text{id}\otimes\text{switch}\otimes\text{id}} & P[A] \otimes P[C] \otimes P[B] \otimes P[D] \\
\uparrow{\scriptstyle \Delta_{A,B}\otimes\Delta_{C,D}} & & \downarrow{\scriptstyle \mu_{A,C}\otimes\mu_{B,D}} \\
P[S] \otimes P[T] \xrightarrow{\mu_{S,T}} & P[I] & \xrightarrow{\Delta_{S',T'}} P[S'] \otimes P[T']
\end{array}$$

Antipode

The antipode of a connected Hopf monoid P consists of the maps

$$\mathrm{s}_I : \mathrm{P}[I] \to \mathrm{P}[I]$$

given by

$$\mathrm{s}_I \;=\; \sum_{\substack{S_1,\dots,S_k \\ k\geq 1}} (-1)^k \, \mu_{S_1,\dots,S_k} \circ \Delta_{S_1,\dots,S_k}.$$

The sum is over all ordered decompositions

$$I = S_1 \sqcup \cdots \sqcup S_k$$

into nonempty disjoint subsets.

$$\mathrm{P}[I] \xrightarrow{\Delta_{S_1,\dots,S_k}} \mathrm{P}[S_1] \otimes \cdots \otimes \mathrm{P}[S_k] \xrightarrow{\mu_{S_1,\dots,S_k}} \mathrm{P}[I]$$

General problem: find an explicit formula for the antipode of a Hopf monoid.

The Hopf monoid of linear orders

The species L of linear orders is a Hopf monoid.
The product is concatenation:

$$\mu_{S,T} : \mathrm{L}[S] \otimes \mathrm{L}[T] \longrightarrow \mathrm{L}[I]$$
$$s_1 \dots s_i \otimes t_1 \dots t_j \longmapsto s_1 \dots s_i \, t_1 \dots t_j$$

The coproduct is restriction:

$$\Delta_{S,T} : \mathrm{L}[I] \longrightarrow \mathrm{L}[S] \otimes \mathrm{L}[T]$$
$$\ell \longmapsto \ell|_S \otimes \ell|_T$$

The antipode is reversal:

$$\mathrm{s}_I : \mathrm{L}[I] \longrightarrow \mathrm{L}[I]$$
$$i_1 i_2 \dots i_n \longmapsto (-1)^n \, i_n \dots i_2 i_1$$

The Hopf monoid of graphs

$\mathrm{G}[I] :=$ vector space with basis the set of graphs with vertex set I.

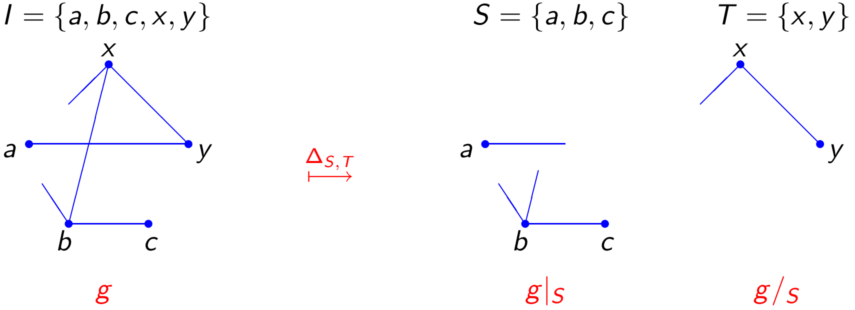
G is a Hopf monoid with

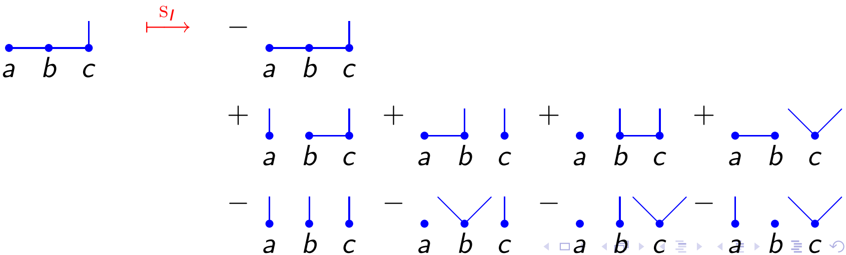
$$\begin{array}{ccc} \mathrm{G}[S] \otimes \mathrm{G}[T] \xrightarrow{\mu_{S,T}} \mathrm{G}[I] & \qquad & \mathrm{G}[I] \xrightarrow{\Delta_{S,T}} \mathrm{G}[S] \otimes \mathrm{G}[T] \\ g_1 \otimes g_2 \longmapsto g_1 \sqcup g_2 & & g \longmapsto g|_S \otimes g/_S \end{array}$$

where

$$\begin{aligned} g_1 \sqcup g_2 &= \text{union of } g_1 \text{ and } g_2, \\ g|_S &= \text{keep everything incident to } S, \\ g/_S &= \text{remove everything incident to } S. \end{aligned}$$

Coproduct and antipode for graphs





The Hopf monoid of matroids

$\mathrm{M}[I] :=$ vector space with basis the set of matroids with ground set I.

M is a Hopf monoid with

$$\begin{aligned} \mathrm{M}[S] \otimes \mathrm{M}[T] &\xrightarrow{\mu_{S,T}} \mathrm{M}[I] \\ m_1 \otimes m_2 &\longmapsto m_1 \oplus m_2 \end{aligned} \qquad \begin{aligned} \mathrm{M}[I] &\xrightarrow{\Delta_{S,T}} \mathrm{M}[S] \otimes \mathrm{M}[T] \\ m &\longmapsto m|_S \otimes m/_S \end{aligned}$$

where

$$\begin{aligned} m_1 \oplus m_2 &= \text{direct sum of } m_1 \text{ and } m_2, \\ m|_S &= \text{restriction of } m \text{ to } S, \\ m/_S &= \text{contraction of } S \text{ from } m. \end{aligned}$$

Characters

Let P be a Hopf monoid. A character ζ consists of maps

$$\zeta_I : \mathrm{P}[I] \to \Bbbk$$

such that for each $I = S \sqcup T$,

$$\begin{array}{ccc} \mathrm{P}[S] \otimes \mathrm{P}[T] & \xrightarrow{\mu_{S,T}} & \mathrm{P}[I] \\ \downarrow{\scriptstyle \zeta_S \otimes \zeta_T} & & \downarrow{\scriptstyle \zeta_I} \\ \Bbbk \otimes \Bbbk & \xrightarrow[\cong]{} & \Bbbk \end{array}$$

and

$$\mathrm{P}[\emptyset] \xrightarrow{\zeta_\emptyset = \mathrm{id}} \Bbbk.$$

Invariants

Let P be a Hopf monoid and ζ a character.
Define, for each $x \in \mathrm{P}[I]$ and $n \in \mathbb{N}$,

$$\chi_I(x)(n) \;:=\; \sum_{S_1 \sqcup \cdots \sqcup S_n = I} (\zeta_{S_1} \otimes \cdots \otimes \zeta_{S_n}) \circ \Delta_{S_1,\ldots,S_n}(x).$$

The sum is over all ordered decompositions of I into n disjoint subsets.

Proposition.

1. $\chi_I(x)$ is a polynomial function of n.
2. $\chi_I(x)(1) = \zeta_I(x)$.
3. $\chi_I(x)(-1) = \zeta_I\Big(\mathrm{s}(x)\Big)$.

The function $\chi_I(x)$ is a polynomial invariant of the structure x (canonically associated to P and ζ).

Polynomial invariants of graphs and matroids

Let $\zeta_I : \mathrm{G}[I] \to \Bbbk$ be

$$\zeta_I(g) \;:=\; \begin{cases} 1 & \text{if } g \text{ consists of half-edges only,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\chi_I(g) =$ chromatic polynomial of g.

Let $\zeta_I : \mathrm{M}[I] \to \Bbbk$ be

$$\zeta_I(m) \;:=\; \begin{cases} 1 & \text{if } m \text{ has a unique basis,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\chi_I(m) =$ polynomial defined by Billera-Jia-Reiner (2006).

Permutahedra

Euclidean space

$$\mathbb{R}^I \ := \ \{\text{functions } x : I \to \mathbb{R}\}.$$

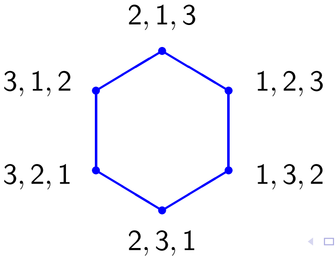
The standard permutahedron

$$\pi_I \ := \ \text{Convex Hull}\{\text{bijective functions } x : I \to [n]\} \subseteq \mathbb{R}^I$$

(where $n = |I|$).

$I \ = \ \{a, b, c\}$ $\{x_a + x_b + x_c = 6\}$

$\pi_I \ =$

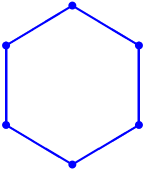


Generalized permutahedra

- Postnikov (2005).
- Postnikov, Reiner, Williams (2007).

Move vertices of π_I keeping new edges parallel to old ones.

The standard permutahedron



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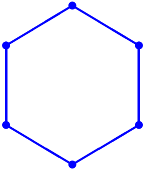

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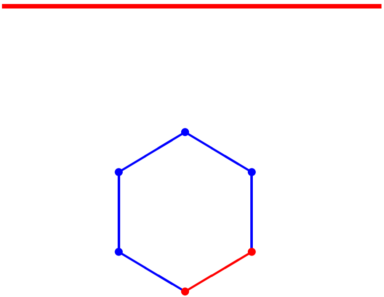

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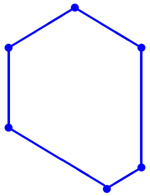
A generalized permutahedron



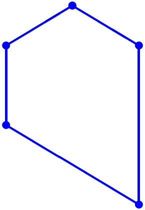
Generalized permutahedra

- Postnikov (2005).
- Postnikov, Reiner, Williams (2007).

Move vertices of π_I keeping new edges parallel to old ones.

Another generalized permutahedron

(the associahedron)



Restriction and contraction for generalized permutahedra

Given $P \subseteq \mathbb{R}^I$ and $v \in \mathbb{R}^I$, let

$$P_v \ := \ \text{subset of } P \text{ where } \langle v, - \rangle \text{ is maximum.}$$

(If P is a polytope, then P_v is a face of P.)

Given $I = S \sqcup T$, let $v_{S,T} \in \mathbb{R}^I$ be any vector such that

$$\begin{cases} v_i = v_j & \text{if } i,j \in S \text{ or } i,j \in T, \\ v_i > v_j & \text{if } i \in S \text{ and } j \in T. \end{cases}$$

Proposition. Let P be a generalized permutahedron.

1. $P_{v_{S,T}}$ depends only on S and T. Let $P_{S,T} := P_{v_{S,T}}$.
2. There are generalized permutahedra $P_1 \subseteq \mathbb{R}^S$ and $P_2 \subseteq \mathbb{R}^T$ such that $P_{S,T} = P_1 \times P_2$. (Note $\mathbb{R}^I = \mathbb{R}^S \times \mathbb{R}^T$.)

Define

$$P|_S \ := \ P_1 \subseteq \mathbb{R}^S, \qquad P/_S \ := \ P_2 \subseteq \mathbb{R}^T.$$

The Hopf monoid of generalized permutahedra

$\mathrm{GP}[I] :=$ vector space with basis the set of generalized permutahedra in $\mathbb{R}^I$.

GP is a Hopf monoid with

$$\mathrm{GP}[S] \otimes \mathrm{GP}[T] \xrightarrow{\mu_{S,T}} \mathrm{GP}[I] \qquad\qquad \mathrm{GP}[I] \xrightarrow{\Delta_{S,T}} \mathrm{GP}[S] \otimes \mathrm{GP}[T]$$

$$P \otimes Q \longmapsto P \times Q \qquad\qquad P \longmapsto P|_S \otimes P/_S$$

(Coassociativity: $P_{R,S,T} = P_1 \times P_2 \times P_3$.)

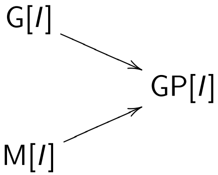
Let $\zeta_I : \mathrm{GP}[I] \to \Bbbk$ be

$$\zeta_I(P) := \begin{cases} 1 & \text{if } P \text{ is a point,} \\ 0 & \text{otherwise.} \end{cases}$$

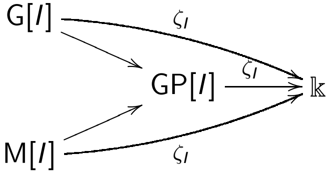
Then $\chi_I(P) =$ polynomial defined by Billera-Jia-Reiner (2006).

Graphic zonotopes, matroid polytopes

There are maps



Proposition. These are morphisms of Hopf monoids.
In addition,



commutes.

The antipode of generalized permutahedra

Theorem. Let P be a generalized permutahedron.

$$\mathrm{S}_I(P) \;=\; \sum_{Q \leq P} (-1)^{|I|+\dim Q}\, Q.$$

The sum is over all faces Q of P.

Proof.

Antipode formula leads to alternating sum of certain faces of P:

$$\mu_{S_1,\ldots,S_k} \circ \Delta_{S_1,\ldots,S_k}(P) \;=\; P_{S_1,\ldots,S_k}.$$

The coefficient of each face is the reduced Euler characteristic of a sphere. □

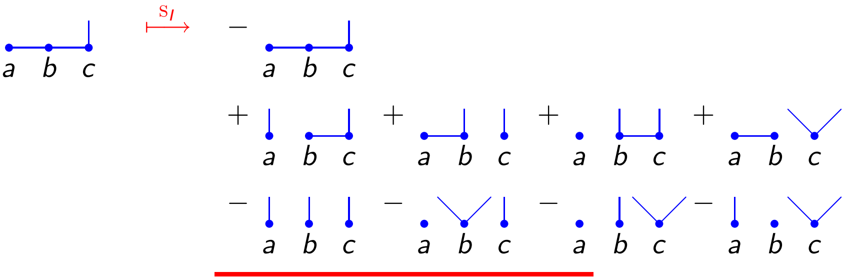
Corollary.

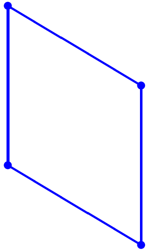
$\chi_I(P)(-1) = (-1)^{|I|}\,\#\{\text{vertices of } P\}$.

$\chi_I(g)(-1) \;= (-1)^{|I|}\,\#\{\text{acyclic orientations of } g\}$ (Stanley).

$\chi_I(m)(-1) = (-1)^{|I|}\,\#\{\text{bases of } m\}$ (Billera-Jia-Reiner).

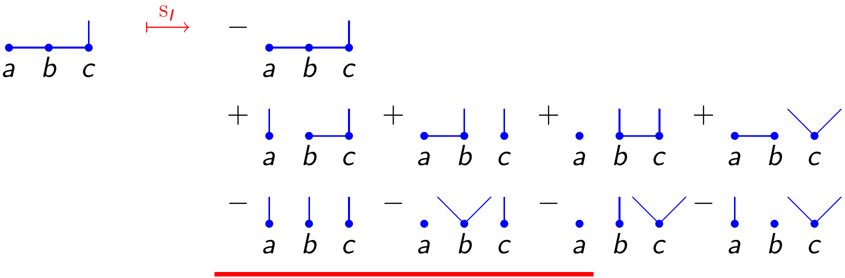
Graphic zonotope and antipode formula

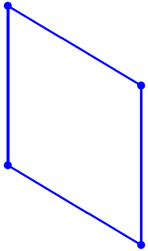




The graphic zonotope
$[a, b] + [b, c] + \{c\}$

Graphic zonotope and antipode formula





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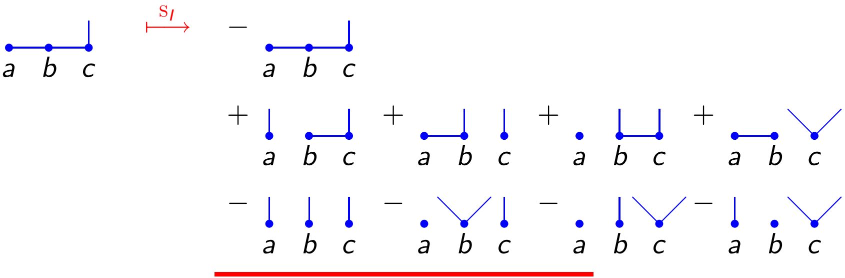

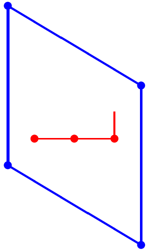

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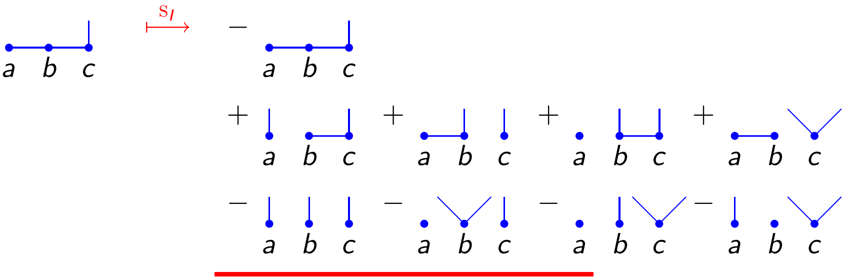

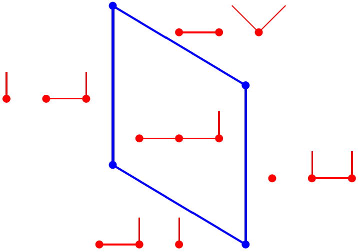

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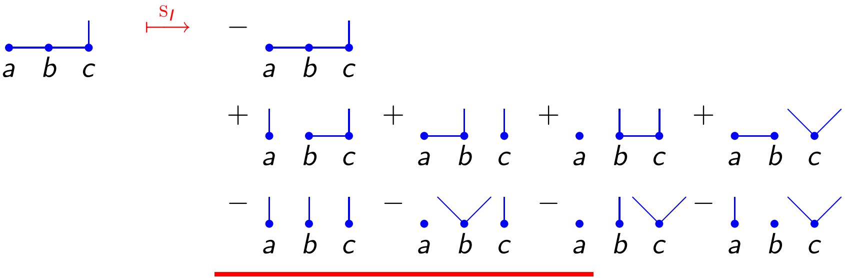
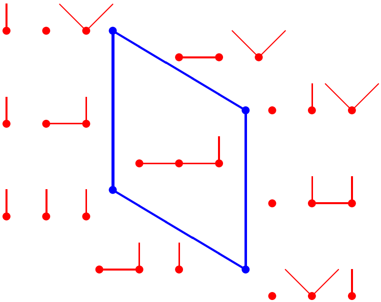
Graphic zonotope and antipode formula





The graphic zonotope
$[a, b] + [b, c] + \{c\}$

From Hopf monoids to Hopf algebras (with Swapneel Mahajan)

number	finite set
graded vector space	species
graded Hopf algebra	Hopf monoid in species

There are constructions that transform Hopf monoids into graded Hopf algebras.
These are bilax monoidal functors

$$\{\text{species}\} \longrightarrow \{\text{graded vector spaces}\}$$

One such is

$$\mathrm{P} \mapsto \bigoplus_{n \geq 0} \mathrm{P}[n]_{S_n}.$$

We have

$$\mathrm{L} \mapsto \Bbbk[x] \quad \text{(polynomials)}$$

$$\mathrm{L} \times \mathrm{L}^* \mapsto \text{Malvenuto-Reutenauer}$$

$$\mathrm{M} \mapsto \text{Schmitt}$$

For more on Hopf monoids, see forthcoming book (with Mahajan)

Monoidal functors, species, and Hopf algebras.

http://www.math.tamu.edu/~maguiar/a.pdf
http://www.math.iitb.ac.in/~swapneel/a.pdf

Thank you.