Pointed Hopf algebras over simple groups

Fernando Fantino

Facultad de Matemática, Astronomía y Física
Universidad Nacional de Córdoba – Argentina

Coloquio de Álgebras de Hopf,
Grupos Cuánticos y Categorías Tensoriales.
La Falda, August 31st, 2009

Joint work with Nicolás Andruskiewitsch, Matías Graña y Leandro Vendramin

On pointed Hopf algebras over some simple groups

Fernando Fantino

Let H be a complex finite-dimensional pointed Hopf algebra. We will show that if the group of group-like elements $G(H)$ is isomorphic to

- an alternating group $\mathbb{A}_m$, $m \geq 5$, or
- a sporadic simple group, different from the Fischer group Fi_{22}, the *Baby monster* B and the *monster* M,

then H is isomorphic to the group algebra $\mathbb{C}G$.

This is part of a joint work with N. Andruskiewitsch, M. Graña and L. Vendramin.


G(H) a simple group.
There is no structure theorem neither for finite groups nor for finite racks.
In order to collect evidence about what groups or what racks might afford finite-dimensional Nichols algebras, it is necessary to study different classes of groups or of racks.

Plan of the talk.

1. **The problem.**
2. **Main results.**
3. **The schemes of the proofs.**

1. The problem.

G finite group and $\Bbbk = \mathbb{C}$.

Classification of finite-dimensional complex pointed Hopf algebras H, with $G(H) \simeq G$.

$\uparrow$ Lifting method (Andruskiewitsch – Schneider)

Determine all *Yetter-Drinfeld modules* W over $\mathbb{C}G$, such that its *Nichols algebra* $\mathfrak{B}(W)$ is finite-dimensional.

W irreducible.

Methods.

$W \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ irreducible $\leftrightarrow$ $(\mathcal{O}, \rho)$, $\mathcal{O}$ conj. class in G, $(\rho, V) \in \widehat{G^\sigma}$, $\sigma \in \mathcal{O}$ fixed

$$(\mathcal{O}, \rho) \rightsquigarrow M(\mathcal{O}, \rho) \rightsquigarrow \mathfrak{B}(\mathcal{O}, \rho)$$

Problem: to discard pairs $(\mathcal{O}, \rho)$ such that $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Strategy: to find a rack $X \subset \mathcal{O}$ such that $(\mathbb{C}X, c^{\mathfrak{q}})$ is a braided vector subspace of $M(\mathcal{O}, \rho)$ and $\dim \mathfrak{B}(\mathbb{C}X, c^{\mathfrak{q}}) = \infty$.

$G(H)$ a simple group.

There is no structure theorem neither for finite groups nor for finite racks.

In order to collect evidence about what groups or what racks might afford finite-dimensional Nichols algebras, it is necessary to study different classes of groups or of racks.

Prominent candidates are the finite simple groups and the finite simple racks. Finite simple racks are classified (Andruskiewitsch – Graña).

For instance, a conjugacy class of a finite simple group is a finite simple rack.

$G(H)$ a simple group.

Let G be a finite simple group.

Question.

Is the group algebra $\mathbb{C}G$ the only (up to isomorphisms) fin.-dim. complex pointed Hopf algebra with group-likes (isomorphic to) G?

i. e.

Does G collapse?

$G(H)$ a simple group.

A finite simple group is isomorphic to one of the following:

- a cyclic group $\mathbb{Z}_p$, of prime order p;
- an alternating group $\mathbb{A}_m$, $m \geq 5$;
- a finite group of Lie type;
- a sporadic group.

Main Theorem

If G is either

(I) the alternating group $\mathbb{A}_m$, $m \geq 5$;

(II) or else a sporadic simple group, different from the Fischer group Fi_{22}, the Baby Monster B, or the Monster M;

then G collapses.

[AFGV1] *Finite-dimensional pointed Hopf algebras with alternating groups are trivial*, `arXiv:0812.4628`.
[AFGV2] *Pointed Hopf algebras over the sporadic simple groups*, `in preparation`.

3. Scheme of the proof.

A basic property of Nichols algebras says that

> if W is a braided subspace of a braided vector space V, then $\mathfrak{B}(W) \hookrightarrow \mathfrak{B}(V)$.

For instance, consider an irreducible $V = M(\mathcal{O}, \rho)$ – say $\dim \rho = 1$ for simplicity.

If X is a proper subrack of $\mathcal{O}$, then $M(\mathcal{O}, \rho)$ has a braided subspace of the form $W = (\mathbb{C}X, c^{\mathfrak{q}})$, which is clearly not a Yetter-Drinfeld submodule but it can be realized as a Yetter-Drinfeld module over smaller groups, that could be reducible if X is decomposable.

If we know that $\dim \mathfrak{B}(X, q) = \infty$, say because we have enough information on one of these smaller groups, then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ too.

The main criterion.

We say that a conjugacy class $\mathcal{O}$ is of *type D* if there exists r and s in $\mathcal{O}$ such that

- $(rs)^2 \neq (sr)^2$.
- r and s are not conjugate in the subgroup $\langle r, s \rangle$.

Theorem

If a conjugacy class $\mathcal{O}$ is of type D, then $\dim \mathfrak{B}(O) = \infty$.

Let $\sigma \in \mathbb{S}_m$ be of type $(1^{n_1}, 2^{n_2}, \dots, m^{n_m})$ and let

$$\mathcal{O} = \begin{cases} (a) \text{ the conjugacy class of } \sigma \text{ in } \mathbb{S}_m, & \text{if } \sigma \notin \mathbb{A}_m, \\ (b) \text{ the conjugacy class of } \sigma \text{ in } \mathbb{A}_m, & \text{if } \sigma \in \mathbb{A}_m. \end{cases}$$

Theorem I

If the type of σ is not in the list below, then $\mathcal{O}$ collapses.

- $(2, 3)$; (2^3); $(1^n, 2)$.
- (3^2); $(2^2, 3)$; $(1^n, 3)$; (2^4); $(1^2, 2^2)$; $(1, 2^2)$.
- (open) $(1, m-1)$, if $m-1$ is prime; (m), if m is prime.

If the type of σ is not:

(i) $(2,3)$; (2^3); $(1^n,2)$;

(ii) (3^2); $(2^2,3)$; $(1^n,3)$; (2^4); $(1^2,2^2)$; $(1,2^2)$;

(iii) (open) $(1, m-1)$, if $m-1$ is prime; (m), if m is prime;

then $\mathcal{O}$ is of type D, hence $\mathcal{O}$ collapses.

For the classes in (ii) and (iii) we look the abelian subracks of $\mathcal{O}$ and use

- if $\mathcal{O}$ is a real conjugacy class (i. e. $x \in \mathcal{O} \Rightarrow x^{-1} \in \mathcal{O}$) of elements with odd order, then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.
- Heckenberger's result for Nichols algebras of b.v.s. of diagonal type.

Then, $\mathbb{A}_m$ collapses for all $m \geq 5$.

Sporadic groups.

There exist 27 sporadic simple groups:

- Mathieu groups: M_{11}, M_{12}, M_{22}, M_{23}, M_{24}.
- Leech lattice groups: HS, J_2, Co_1, Co_2, Co_3, McL, Suz.
- Pariahs: J_1, ON, J_3, Ru, Ly, J_4.
- Monster sections: He, HN, Th, Fi_{22}, Fi_{23}, Fi'_{24}, B, M.
- Miscellaneous: T.

Theorem II.

If G is a sporadic simple group and $\mathcal{O}$ is a non-trivial conjugacy class of G NOT listed in the Tables below, then $\mathcal{O}$ is of type D. Hence $\mathcal{O}$ collapses.

G	# **Classes**	**Classes**
M_{11}	10	8A, 8B, 11A, 11B
M_{12}	15	11A, 11B
M_{22}	12	11A, 11B
M_{23}	17	23A, 23B
M_{24}	26	23A, 23B
J_2	21	2A, 3A
Suz	43	3A
HS	24	11A, 11B
McL	24	11A, 11B
Co_3	42	23A, 23B

Sporadic groups.

G	# **Classes**	**Classes**
Co_2	60	2A, 23A, 23B
Co_1	101	23A, 23B, 33A
J_1	15	15A, 15B, 19A, 19B, 19C
$O'N$	30	31A, 31B
J_3	21	5A, 5B, 19A, 19B
Ru	36	29A, 29B
He	33	all collapse
Fi_{22}	65	2A, 22A, 22B
Fi_{23}	98	2A, 23A, 23B
HN	54	all collapse
Th	48	3A, 31A, 31B
T	22	2B

Sporadic groups.

G	# **Classes**	**Classes**
Ly	53	33A, 33B, 37A, 37B, 67A, 67B, 67C
J_4	62	29A, 37A, 37B, 37C, 43A, 43B, 43C
Fi'_{24}	108	9D, 23A, 23D, 27A, 27B, 27C,
		29A, 29B, 39A, 39B, 39C, 39D
B	184	2A, 2C, 16C, 16D, 32A, 32B, 32C, 32D, 34A,
		40E, 46A, 46B, 47A, 47B, 60C
M	194	32A, 32B, 33A, 46A, 46B, 46C, 47A, 47B,
		66B, 69A, 69B, 87A, 87B, 93A, 93B, 94A, 94B

Comments.

We use another techniques to prove that the remaining classes $\mathcal{O}$ give rise to infinite-dimensional $\mathfrak{B}(\mathcal{O}, \rho)$.

The Fischer group Fi_{22}, and the monster groups B and M are almost completed.

We are working in the same problem for finite groups of Lie type.

References of computational tools

- ATLAS of Finite Groups Representations
  It containts information on about 715 groups
- GAP 4.4.12
  A System for Computational Discrete Algebra
- AtlasRep 1.4.0
  A GAP Interface to the ATLAS
  http://www.gap-system.org/Packages/