COMPACT INVOLUTIONS OF SEMISIMPLE QUANTUM GROUPS *) **)

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It is proved that a complex cosemisimple Hopf algebra has at most one compact involution modulo automorphisms.

Contents

Introduction

Let H be a complex cosemisimple Hopf algebra, that is, any finite dimensional H-comodule is completely reducible, or equivalently H is completely reducible as comodule via the comultiplication (see 1.3 (c) in [1]). We prove that two compact involutions of H [2] are necessarily conjugated by a Hopf algebra automorphism. This extends a well-known theorem of Cartan to the quantum case. Using results from [3], this was proved recently for finite Hopf algebras [4]. Since then, the author noticed however the paper [5] which contains a weak form of those results from [3] and enables him to extend the theorem to the infinite case. The second part of the proof is a variation of Mostow's proof of the above mentioned Cartan's theorem — see p. 182 in [6]. In the first section of this paper, we recall some results on cosemisimple Hopf algebras (some of them go back to [7]) and give a formula (1.8) for the Killing form — an invariant bilinear form on H arising from (a choice of) the integral and normalized by a further invariant condition. In the second, we prove the theorem. For this, we use an invariant sesquilinear form on H also derived from the integral, first considered in [8].

1 Killing forms on cosemisimple Hopf algebras

We shall work over an arbitrary field $\mathbb{K}$ in this section. The notation for Hopf algebras is standard: Δ, S, ε, denote respectively the comultiplication, the antipode, the counit; we use Sweedler [9] notation but drop the summatory.

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1.1. Let H be a Hopf algebra. Recall that for a finite dimensional right comodule $c : V \to V \otimes H$, its left and right duals ${}^d c$ and c^d are the right H-comodule structures on V^* defined as follows. Let $(h_i)_{i \in I}$ be a basis of H. Then $c(v) = \sum_i T_i(v) \otimes h_i$, with $T_i \in \operatorname{End} V$, $T_i = 0$ for all but a finite number of i. Define

$$ {}^d c(\alpha) = \sum_i {}^t T_i(\alpha) \otimes \mathcal{S}^{-1}(h_i), \qquad c^d(\alpha) = \sum_i {}^t T_i(\alpha) \otimes \mathcal{S}(h_i), $$

for $\alpha \in V^*$. ${}^d V$, V^d denote V^* considered as H-comodule via, respectively, ${}^d c$, c^d. In the category of finite dimensional right comodules, the functors $V \mapsto {}^d V$ and $V \mapsto V^d$ are inverse to each other; therefore, the following are equivalent:

$$ \text{(a)} \quad V \simeq (V^d)^d; \quad \text{(b)} \quad V \simeq {}^d({}^d V); \quad \text{(c)} \quad V^d \simeq {}^d V. $$

1.2. H^* has an algebra structure provided by the transposes of the multiplication and the counit. Any (left or right) H-comodule is then a (right or left) H^*-module; such H^*-modules are called rational. For example, H is an H^*-bimodule via

$$ x \rightharpoonup h = h_{(1)} \langle x, h_{(2)} \rangle, \quad h \leftharpoonup x = \langle x, h_{(1)} \rangle h_{(2)}; \qquad h \in H, x \in H^*. $$

This correspondence is in fact an isomorphism between the categories of H-comodules and rational H^*-comodules. By psychological reasons, it is often helpful to state properties in terms of H^*-actions. By abuse of notation, we write $\mathcal{S} : H^* \to H^*$ for the transpose of the antipode and $\varepsilon : H^* \to \mathbb{K}$ for evaluation in 1. The representations ρ^d and ${}^d\rho$ can be defined for any representation ρ of H^*; for rational ones, they agree with those derived from the previous c^d, ${}^d c$.

1.3. Define $\psi^V : (\operatorname{End} V)^* \to H$ by $\psi^V(\alpha) = \sum_i \langle \alpha, T_i \rangle h_i$. Then ψ^V is a morphism of coalgebras. Furthermore, it is injective if V is irreducible, and the simple subcoalgebras of H are exactly the $\operatorname{Im} \psi^V$ for V irreducible [1]. Thus, if H is cosemisimple,

$$ H = \oplus_{V \in \widehat{H}} \operatorname{Im} \psi^V, $$

where $\widehat{H}$ denotes the set of isomorphism classes of irreducible H comodules. (We often confuse a class with a representant). $\operatorname{Im} \psi^V$ is the isotypic component of H, for the coaction given by the multiplication, of type V. We shall denote it alternatively as H_c or H_ρ; ρ will be then the representation of H^* derived from the coaction c. We shall also identify $\widehat{H}$ with the set of isomorphism classes of irreducible rational H^*-modules.

Given a finite dimensional representation $\rho : H^* \to \operatorname{End} U$, let $\phi^U : U^* \otimes U \to H^{**}$ be the "matrix coefficient" map defined, for $v \in U$, $\alpha \in U^*$, by $\langle \phi^U_{\alpha \otimes v}, x \rangle = \langle \alpha, \rho(x) v \rangle$. Modulo the usual identifications $(\operatorname{End} U)^* \simeq \operatorname{End} U$ (provided by the trace) and $\operatorname{End} U \simeq U^* \otimes U$, it coincides with the usual transpose map ${}^t\rho : (\operatorname{End} U)^* \to H^{**}$:

$$ {}^t\rho(T) = \phi^U_{\alpha \otimes v}, \quad \text{if} \quad T \in \operatorname{End} U, \quad T(u) = \langle \alpha, u \rangle v. $$

Note that ${}^tS\left(\phi^V_{\alpha\otimes v}\right) = \phi^{U^d}_{v\otimes\alpha}$. Let $\Theta : H \to H^{**}$ be the natural injection; then $\Theta\psi^V = \phi^V$ (V is an H-comodule and hence a rational H^*-module). Θ is a morphism of H^*-bimodules.

1.4. Let $d : W \to W \otimes H$ be another finite dimensional right comodule structure; then $V \otimes W$ also is an H-comodule whose coaction we shall denote $c \otimes d$. Let $S_j \in \operatorname{End} W$ be, similarly as above, such that $d(w) = \sum S_j(w) \otimes h_j$. Define a comodule structure on $\operatorname{Hom}(V, W)$ by $A \mapsto \sum_{i,j} S_j \circ A \circ T_i \otimes h_j \mathcal{S}(h_i)$. The natural isomorphism between $\operatorname{Hom}(V, W)$ and $W \otimes V^*$ is in fact an H-comodule isomorphism between $\operatorname{Hom}(V, W)$ and $W \otimes V^d$. The isotypic component of trivial type of $\operatorname{Hom}(V, W)$ with respect to the adjoint action is exactly the space of H-comodule maps. Therefore, if W and V are irreducible, the multiplicity of the trivial representation in $W \otimes V^d$ is 1 (resp., 0) if W and V are (resp., are not) isomorphic. In other words, $W \otimes V$ contains the trivial representation if and only if $W \simeq {}^dV$.

1.5. Recall that a linear functional $\int : H \to \mathbb{K}$ is a *right integral* if

$$\langle \textstyle\int, h\rangle 1 = \langle \textstyle\int, h_{(1)}\rangle h_{(2)}, \qquad \text{forall}\, h \in H. \tag{1.1}$$

It is equivalent to provide [10]

(a) A right integral $\int$.

(b) A bilinear form $((\,|\,)) : H \times H \to \mathbb{K}$ satisfying

$$\begin{aligned} ((uv|w)) &= ((u|vw)), && (1.2)\\ ((x \rightharpoonup v|w)) &= ((v|\mathcal{S}x \rightharpoonup w)), && (1.3) \end{aligned}$$

for all $u, v, w \in H$, $x \in H^*$.

Explicitly, $\langle \int, v\rangle = ((v|1))$, $((u|v)) = \langle \int, uv\rangle$. In general, if $(\,|\,)$ is a bilinear form which satisfies (1.3), then $\Lambda \in H^*$ given by $\langle \Lambda, v\rangle = (v|1)$ is a right integral; (1.2) is a "normalization" condition which ensures the bijectivity of the correspondence. Indeed, if $(\,|\,)$ satisfies (1.3) then $((u|v)) = (uv|1)$ also does, and in addition satisfies (1.2).

Now let $M, N \subseteq H$ be submodules for $\rightharpoonup$ and let $\theta : M \to N^d$ be given by $\langle\theta(m), n\rangle = ((m|n))$; θ is a morphism of A-modules by (1.3). Therefore if M and N are both irreducible, θ is either 0 or an isomorphism. Taking $M = \mathbb{K}1 = H_\varepsilon$, the trivial submodule of H, we conclude that $\langle \int, v\rangle = 0$ for all $v \in N$, for all irreducible, non-trivial, N.

Now assume that H is cosemisimple. For $a \in H$, write $a = \sum_{\rho\in\widehat{H}} a_\rho$, with $a_\rho \in H_\rho$. By abuse of notation, we shall write $a_\varepsilon.1$ instead of a_ε with $a_\varepsilon \in \mathbb{K}$. Then

$$\langle \textstyle\int, h\rangle = a_\varepsilon \langle \textstyle\int, 1\rangle. \tag{1.4}$$

Conversely, the linear map defined by (1.4) and an arbitrary value of $\langle \int, 1\rangle$ is a right integral, because H_ρ is a subcoalgebra of H. It follows that, for H cosemisimple, the space of right integrals is one-dimensional. Interchanging right by left and viceversa, one sees that any left integral also is expressed by (1.4); hence H is unimodular. In particular, by the "dual hand" version of the equivalence above, $((\,|\,))$ also satisfies

$$((v \leftharpoonup \mathcal{S}x|w)) = ((v|w \leftharpoonup x)). \tag{1.5}$$

Finally, if H is an arbitrary Hopf algebra admitting a right integral such that $\langle \int, 1\rangle \neq 0$ then H is cosemisimple. See [7], where the formula (1.4) appears for the first time.

Lemma 1.6. *Let H, H' be Hopf algebras, let $T : H' \to H$ be an isomorphism of coalgebras such that $T(1) = 1$ and let $\int$ be a right integral for H. Then $\int \circ T$ is a right integral for H'. In particular, $\int \circ \mathcal{S}$ is a left integral for H. If H is cosemisimple, T is an automorphism of Hopf algebras of H and $\int$ is normalized by $\langle \int, 1\rangle = 1$, then $((Tu|Tv)) = ((u|v))$, for all $u, v \in H$.*

Proof. Straightforward. □

1.7. Let H be a cosemisimple Hopf algebra as above.

Theorem (Thm. 3.3 in [5]). *For each simple subcoalgebra C of H, $\mathcal{S}^2C = C$.*

Corollary. *For any irreducible H-comodule c, c^{dd} is isomorphic to c.*

Proof. Let V be the space of c. Then $\mathcal{S}^2\left(\phi^V_{\alpha\otimes v}\right) = \phi^{V^{dd}}_{\alpha\otimes v} \in H_c \cap H_{c^{dd}}$ (modulo identification by Θ). Thus $H_c = H_{c^{dd}}$ and hence $c \simeq c^{dd}$. □

As observed in [5], the proof of this theorem implies that $((\,|\,))$ is non-degenerate. This fact will also follow from formula (1.8) below.

1.8. We still assume that H is cosemisimple and normalize $\int$ by $\langle \int, 1\rangle = 1$. The corresponding $((\,|\,))$ will be named the Killing form of H. We shall give a formula for it in the spirit of [3]. Let $a = \sum_{c\in\widehat{H}} a_c$, $b = \sum_{c\in\widehat{H}} b_c \in H$. Then

$$((a|b)) = \sum_{c\in\widehat{H}} ((a_{c^d}|b_c)).$$

So we need only to precise $((\,|\,)) : H_{c^d} \otimes H_c \to \mathbb{K}$, for $c : V \to V \otimes H$ irreducible. Recall that we have identified $H_c \simeq (\operatorname{End} V)^*$ with $\operatorname{End} V$ via the trace map. Fix $\mathcal{M} \in \operatorname{Aut} V$ such that

$$\sum_i T_i\mathcal{M} \otimes h_i = \sum_i \mathcal{M}T_i \otimes \mathcal{S}^2(h_i). \tag{1.6}$$

Let $\rho : H^* \to \operatorname{End} V$ be the representation corresponding to c. Then (1.6) means that $\mathcal{M}\rho(\mathcal{S}^2x) = \rho(x)\mathcal{M}$, for all $x \in H^*$. Let $S \in \operatorname{End}(V^d)$, $T \in \operatorname{End} V$ and define

$$B_c(S,T) = \operatorname{Tr}({}^tST\mathcal{M}). \tag{1.7}$$

Then

$$\begin{aligned} B_c(x \rightharpoonup S, T) &= \operatorname{Tr}({}^t(\rho^d(x)S)T\mathcal{M}) = \operatorname{Tr}({}^tS\,{}^t(\rho^d(x))T\mathcal{M}) = \\ &= \operatorname{Tr}({}^tS\rho(\mathcal{S}x)T\mathcal{M}) = B_c(S, \mathcal{S}x \rightharpoonup T)\,. \end{aligned}$$

On the other hand,

$$\begin{aligned} B_c(S \leftharpoonup \mathcal{S}x, T) &= \mathrm{Tr}\,({}^t(S\rho^d(\mathcal{S}x))T\mathcal{M}) = \mathrm{Tr}\,({}^t(\rho^d(\mathcal{S}x))^t ST\mathcal{M}) = \\ &= \mathrm{Tr}\,({}^tST\mathcal{M}^t(\rho^d(\mathcal{S}x))) = \mathrm{Tr}\,({}^tST\mathcal{M}\rho(\mathcal{S}^2x)) = \\ &= \mathrm{Tr}\,({}^tST\rho(x)\mathcal{M}) = B_c(S, T \leftharpoonup x)\,. \end{aligned}$$

As End V is irreducible as H^*-bimodule, there is only one bilinear form satisfying (1.3) and (1.5), up to scalars. Therefore,

$$((a_{c^d}|b_c)) = C_c B_c(S,T) = C_c \mathrm{Tr}\,({}^tST\mathcal{M}),$$

for some scalar C_c, where $S \in \mathrm{End}\,(V^d)$ corresponds to a_{c^d}, and T to b_c. Next we compute C_c. The preceding $B_c(\, , \,)$ depends on $\mathcal{M}$ and hence is also defined up to a scalar; what we need, therefore, is to take $C_c = 1$ and adjust $\mathcal{M}$.

So let a_{ρ^d}, b_ρ, S and T be as above. We wish to compute $((a_{\rho^d}|b_\rho)) = ((a_{\rho^d}b_\rho|1))$ $= d_\epsilon$, if $a_{\rho^d}b_\rho = \sum_{\tau\in\widehat{H}} d_\tau$, with $d_\tau \in H_\tau$ and $d_\epsilon.1$, $d_\epsilon \in \mathbb{K}$, instead of d_ϵ. We compute $a_{\rho^d}b_\rho$ (compare with [11]). $V^d \otimes V$ decomposes as direct sum of irreducible A-submodules: $V^d \otimes V = \oplus_{\tau\in J} U_\tau$. Let $\iota_\tau : U_\tau \to V^d \otimes V$ be the inclusion and $\pi_\tau : V^d \otimes V \to U_\tau$, the projection with respect to this direct sum. Let $R_{\tau\mu} = \pi_\mu(S \otimes T)\iota_\tau \in \mathrm{Hom}(U_\tau, U_\mu)$. Then $S \otimes T = \sum_{\tau,\mu} \iota_\mu R_{\tau\mu}\pi_\tau$; that is, $(R_{\tau\mu})$ is the "partition" of $S \otimes T$ in blocks with respect to the decomposition above, and d_ϵ corresponds to $R_{\epsilon\epsilon}$. We already know that $(V^d \otimes V)_\epsilon$ is one dimensional. A generator is $Z = \sum_{1\le h\le n} \alpha_h \otimes \mathcal{M}v_h$, where (v_h) is a basis of V and (α_h) is the dual basis. Indeed,

$$\begin{aligned} (c^d \otimes c)(Z) &= \sum_{1\le h\le n, i,j\in I} {}^tT_j(\alpha_h) \otimes T_i(\mathcal{M}v_h) \otimes \mathcal{S}(h_j)h_i \\ &= \sum_{1\le h,k\le n, i,j\in I} \langle v_k, {}^tT_j(\alpha_h)\rangle \alpha_k \otimes T_i(\mathcal{M}v_h) \otimes \mathcal{S}(h_j)h_i \\ &= \sum_{1\le k\le n, i,j\in I} \alpha_k \otimes T_i(\mathcal{M}T_j v_k) \otimes \mathcal{S}(h_j)h_i \\ &= \sum_{1\le k\le n, i,j\in I} \alpha_k \otimes T_iT_j\mathcal{M}(v_k) \otimes \mathcal{S}^{-1}(h_j)h_i = Z \otimes 1\,. \end{aligned}$$

Now the projector $\pi_\epsilon : V^d \otimes V \to \mathbb{K}Z$ must be of the form $\pi_\epsilon(P) = \langle \Omega, P\rangle Z$, for $P \in V^d \otimes V$, with $\Omega \in (V^d \otimes V)^*$. Let $\Omega = \sum_{1\le i\le n} v_i \otimes \alpha_i$ (with the usual vector space identification of $(V^d \otimes V)^*$ with $V \otimes V^d$) and write tentatively π for $P \mapsto \langle \Omega, P\rangle Z$. Then $c_{\mathrm{Hom}(V^d\otimes V, \mathbb{K}Z)}(\pi) = \sum_{i,j\in I} \mathrm{id} \circ \pi \circ ({}^tT_i \otimes T_j) \otimes \mathcal{S}\,(\mathcal{S}(h_i)h_j)$. Evaluating in $\beta \otimes w$ the first factor, we get

$$\sum_{i,j\in I} \langle \Omega, {}^tT_i(\beta) \otimes T_j(w)\rangle Z \,\otimes\, \mathcal{S}\,(\mathcal{S}(h_i)h_j)$$

$$= \sum_{\substack{1\le k\le n \\ i,j\in I}} \langle v_k, {}^tT_i(\beta)\rangle\langle \alpha_k, T_j(w)\rangle Z \otimes \mathcal{S}\,(\mathcal{S}(h_i)h_j) =$$

$$
\begin{aligned}
&= \sum_{i,j\in I} \langle \beta, T_iT_j(w)\rangle Z \otimes \mathcal{S}\left(\mathcal{S}(h_i)h_j\right) \\
&= \sum_{i\in I} \langle \beta, T_i(w)\rangle Z \otimes \mathcal{S}\left(\mathcal{S}(h_{i(1)})h_{i(2)}\right) \\
&= \langle \beta, w\rangle Z \otimes 1 = \langle \Omega, \beta \otimes w\rangle Z \otimes 1;
\end{aligned}
$$

that is, π is invariant, and nonzero. As some multiple of it is a projector, $\pi(Z) = \langle \Omega, Z\rangle Z = \operatorname{Tr} Z \neq 0$. Therefore, we can normalize $\mathcal{M}$, as promised, by $\operatorname{Tr}\mathcal{M} = 1$. We can now write π_ϵ instead of π. But $d_\epsilon Z = \pi_\epsilon((S\otimes T)Z) = \langle \Omega, (S\otimes T)Z\rangle Z$ and hence

$$
\begin{aligned}
d_\epsilon = \langle \Omega, (S\otimes T)Z\rangle &= \left\langle \sum_i v_i \otimes \alpha_i, \sum_j S\alpha_j \otimes T\mathcal{M}v_j \right\rangle \\
&= \sum_{i,j} \langle \alpha_i, T\mathcal{M}v_j\rangle \langle \alpha_j, {}^tSv_i\rangle = \operatorname{Tr}({}^tST\mathcal{M}).
\end{aligned}
$$

We have proved

$$
((a_{\rho^d} | b_\rho)) = \operatorname{Tr}({}^tST\mathcal{M}), \tag{1.8}
$$

where a_{ρ^d} corresponds to $S \in \operatorname{End}(V^d)$, b_ρ to T and $\mathcal{M} \in \operatorname{End} V$ satisfies (1.6) and $\operatorname{Tr}\mathcal{M} = 1$.

1.10. Is the Killing form symmetric? We compute $((b_\rho | a_{\rho^d})) = ((b_{\tau^d} | a_\tau))$, for $\tau = \rho^d$. Note that (1.6) is equivalent to

$$
({}^t\mathcal{M})^{-1}\rho^d(\mathcal{S}^2x) = \rho^d(x)({}^t\mathcal{M})^{-1}, \qquad \text{for all } x \in H^*.
$$

Also, if b_ρ corresponds to $T \in \operatorname{End} V$ then it corresponds to $\mathcal{M}^{-1}T\mathcal{M} \in \operatorname{End} V^{dd}$. Let $\mu = (\operatorname{Tr}(M^{-1}))^{-1}$. Applying (1.8) to ρ^d we get

$$
\begin{aligned}
((b_\rho | a_{\rho^d})) &= \mu \operatorname{Tr}({}^t(\mathcal{M}^{-1}T\mathcal{M})S({}^t\mathcal{M})^{-1}) = \\
&= \mu \operatorname{Tr}({}^tT({}^t\mathcal{M})^{-1}S) = \mu \operatorname{Tr}({}^tS\mathcal{M}^{-1}T).
\end{aligned}
$$

Thus the Killing form is symmetric if and only if $\mathcal{M} = (\dim V)^{-1}\mathrm{id}_V$ for all irreducible V, if and only if $\mathcal{S}^2 = \mathrm{id}$. Indeed, $\mathcal{S}^2 b_\rho$ corresponds to $\mathcal{M}T\mathcal{M}^{-1} \in \operatorname{End} V$.

2 Killing forms and *-Hopf algebras

We assume in this section that $\mathbb{K} = \mathbb{C}$. We suppose further that H is a *-Hopf algebra, i.e., it is a *-algebra and the comultiplication is a morphism of *-algebras; H^* is then considered as *-algebra by $\langle x^*, v\rangle = \overline{\langle x, \mathcal{S}(v)^*\rangle}$. It is known that $(\mathcal{S}x)^* = \mathcal{S}^{-1}(x^*)$. For convenience, we shall denote $T(x) = (\mathcal{S}x)^* = \mathcal{S}^{-1}(x^*)$.

Lemma 2.1. *(i) The following data are equivalent:*

(a) A right integral $\int : H \to \mathbb{C}$.

(b) A bilinear form $((\,|\,))$ *satisfying (1.2), (1.3).*

(c) A sesquilinear form $(\,|\,)_\ell$ *satisfying*

$$(uv|w)_\ell = (v|u^*w)_\ell\,, \tag{2.1}$$

$$(x \rightharpoonup v|w)_\ell = (v|x^* \rightharpoonup w)_\ell\,. \tag{2.2}$$

(ii) Also, the following are equivalent:

(d) A left integral $\int : H \to \mathbb{C}$.

(e) A bilinear form $((\,|\,))_r$ *satisfying (1.2), (1.6).*

(f) A sesquilinear form $(\,|\,)_r$ *satisfying*

$$(uv|w)_r = (u|v^*w)_r \tag{2.3}$$

$$(v \leftharpoonup x|w)_r = (v|w \leftharpoonup x^*)_r\,. \tag{2.4}$$

Proof. We have already discussed the equivalence between (a) and (b), resp. (d) and (e). The correspondence between (b) and (c), resp. (e) and (f), is given by

$$(v|w)_\ell = ((w^*|v)), \qquad \text{resp.} \quad (v|w)_r = ((v^*|w))_r\,, \tag{2.5}$$

and correspondingly, $((v|w)) = (w|v^*)_\ell$, $((v|w))_r = (v^*|w)_r$. For the proof, we need the formulas

$$(x \rightharpoonup v)^* = (Sx)^* \rightharpoonup v^*\,, \qquad (v \leftharpoonup x)^* = v^* \leftharpoonup (Sx)^*\,.$$

Thus $(v|x^* \rightharpoonup w)_\ell = (((x^* \rightharpoonup w)^*|v)) = ((S^{-1}x \rightharpoonup w^*|v)) = ((w^*|x \rightharpoonup v)) = (x \rightharpoonup v|w)_\ell$, and the rest is similar. □

2.2. Let $\int$ be a right integral and let Λ be defined by $\langle \Lambda, h\rangle = \overline{\langle \int, h^*\rangle}$. Then Λ is also a right integral:

$$\langle \Lambda, h_{(1)}\rangle h_{(2)} = \overline{\langle \textstyle\int, h_{(1)}{}^*\rangle} h_{(2)} = \left(\langle \textstyle\int, h_{(1)}{}^*\rangle h_{(2)}{}^*\right)^* = \left(\langle \textstyle\int, h^*\rangle 1\right)^* = \langle \Lambda, h\rangle 1\,.$$

Assume now that H is cosemisimple. We shall normalize, in what follows, $\int$ by $\langle \int, 1\rangle = 1$. Then, by the uniqueness of the right integral, $\int = \Lambda$. It follows that the corresponding sesquilinear form $(\,|\,)_\ell$ is Hermitian:

$$(v|w)_\ell = \langle \textstyle\int, w^*v\rangle = \langle \Lambda, w^*v\rangle = \overline{\langle \textstyle\int, (w^*v)^*\rangle} = \overline{(w|v)_\ell}\,.$$

Remark. These facts were essentially first observed by Majid [8].

2.3. A $*$-representation of H^* is a representation $\rho : H^* \to \operatorname{End} V$ together with a non-degenerate sesquilinear form $(\,|\,)$ such that $(\rho(x)v|w) = (v|\rho(x^*)w)$, for all $x \in H^*$, $v, w \in V$. Such form shall be called invariant. We consider in the following only finite dimensional rational representations. A representation is a

*-representation if and only if there exists a sesquilinear isomorphism $J : V \to V^d$ such that $J(\rho(x)w) = \rho^d(Tx)J(w)$. Explicitly, $\langle Jw, v\rangle = (v|w)$. If $T \in \operatorname{End} V$, define as usual $T^* \in \operatorname{End} V$ by $(Tv|w) = (v|T^* w)$, or equivalently by $T^* = J^{-1}{}^tTJ$.

Let V be a right H-comodule and let T_i as in **1.1**. Let $\mathfrak{S} = \sum_i T_i \otimes h_i$; it follows easily from the comodule axioms that $\mathfrak{S}$ is inversible and $\mathfrak{S}^{-1} = \sum_i T_i \otimes \mathcal{S}(h_i)$, in the algebra $\operatorname{End} V \otimes H$. The last is a *-algebra once a non-degenerate sesquilinear form is chosen. It can be shown that the corresponding rational representation of H^* is a *-representation if and only if $\mathfrak{S}^{-1} = \mathfrak{S}^*$: hence the present definition agrees with that of [2].

Let V be a *-representation. Let $(J^{-1})^\dagger : V^* \to V$ be given by $\langle \mu, (J^{-1})^\dagger\alpha\rangle = \overline{\langle \alpha, J^{-1}\mu\rangle}$. Then the * in H of the matrix coefficients is given (modulo Θ) by [11], p. 306

$$\phi^{V}_{\alpha\otimes v}{}^{*} = \phi^{V^d}_{(J^{-1})^\dagger\alpha\otimes Jv}\,. \tag{2.6}$$

Equivalently, if $T \in \operatorname{End} V$ corresponds to $w \in H$, then w^* corresponds to

$$JTJ^{-1} \in \operatorname{End} V^d\,. \tag{2.7}$$

Here one uses that $\operatorname{Tr}(JAJ^{-1}) = \overline{\operatorname{Tr} A}$, for $A \in \operatorname{End} V$.

If $(\,|\,)$ is an invariant form, then $(\,|\,)_{\mathrm{opp}}$, given by $(v|w)_{\mathrm{opp}} = \overline{(w|v)}$, also is. Assume that V is irreducible. Then invariant forms are unique up to multiplication of a scalar; in particular $(\,|\,)_{\mathrm{opp}} = \lambda(\,|\,)$ for some scalar λ. Applying this twice, we see that $\lambda\overline{\lambda} = 1$. Multiplying $(\,|\,)$ by a suitable scalar, we can assume that $\lambda = 1$, i.e., that $(\,|\,)$ is Hermitian.

Let V be a *-representation, with invariant form $(\,|\,)$, and let $\mathcal{M} \in \operatorname{Aut} V$ satisfying (1.6). Let $(\,|\,)_d$ be the form on V^d defined by $(\mu|\eta)_d = (\mathcal{M}^{-1}J^{-1}\eta|J^{-1}\mu)$; it is also invariant. If V is irreducible, then V^d also is; assuming this, we shall normalize first $(\,|\,)$ to get an Hermitian form, and second $\mathcal{M}$, to get an Hermitian form on V^d. In such case, $\mathcal{M} = \mathcal{M}^*$, i.e., $\mathcal{M}$ is self-adjoint. Now asume in addition that $(\,|\,)$ is an inner product. Then $(\,|\,)_d$ also is, if and only if $\mathcal{M}$ is positive definite; in such case, $\operatorname{Tr} \mathcal{M} > 0$. Conversely, if V^d admits an invariant inner product, then some multiple of $\mathcal{M}$ is positive definite.

A representation is not always a *-representation. For example, let H^* be the group algebra of an abelian finite group with the involution $(\sum_{g\in G} \lambda_g e_g)^* = \sum_{g\in G} \overline{\lambda_g} e_g$. Let χ be a one-dimensional representation of G which is not real; this admits no sesquilinear invariant form.

2.4. Now we are ready to state the key point of the proof of the main result. We first recall a definition [2].

Definition. We shall say that H is a *compact quantum group* if any rational, finite dimensional, representation of H^* carries an invariant inner product.

By a standard argument, if H is compact, then is cosemisimple. It is known (see e.g. [12], [13]) that completions of compact quantum groups as in the preceding definition with respect to a suitable norm give rise to compact quantum groups as in [2]; the preceding notion corresponds to that of "algebras of regular functions" in Woronowicz definition [2].

Proposition. *H is a compact quantum group if and only if the hermitian form $(\,|\,)_\ell$ is positive defined.*

Proof. If $(\,|\,)_\ell$ is positive defined then any H^*-submodule of H (for $\rightharpoonup$) carries an invariant inner product and H is a compact quantum group. Conversely, assume that H is a compact quantum group. Let $v \in H_\rho$, $w \in H_\tau$; then $w^* \in H_{\tau^*}$ by (2.6), and $(v|w)_\ell = 0$ if ρ and τ are not isomorphic, by (2.5). So assume that $\rho = \tau$ and let $S, T \in \operatorname{End} V$ correspond to v, w, respectively. By (1.7) and (2.7), we have

$$\begin{aligned}(v|w)_\ell = ((w^*|v)) &= \operatorname{Tr}\,({}^t(JTJ^{-1})S\mathcal{M}) = \operatorname{Tr}\,({}^t\mathcal{M}{}^tSJTJ^{-1}) \\ &= \operatorname{Tr}\,(J\mathcal{M}S^*TJ^{-1}) = \overline{\operatorname{Tr}\,(\mathcal{M}S^*T)} = \operatorname{Tr}\,(T^*S\mathcal{M})\end{aligned}$$

(This formula also implies that $(\,|\,)_\ell$ is Hermitian). Thus $(v|v)_\ell = \operatorname{Tr}(S^*S\mathcal{M}) > 0$ if $S \neq 0$, because $\mathcal{M}$, normalized by $\operatorname{Tr}\mathcal{M} = 1$, is positive definite. □

2.5. The preceding Proposition enables us to adapt Mostow's proof of Cartan's theorem of the uniqueness of compact involutions (see Ch. II, Thm. 7.1 in [6]) to our setting. See also Proposition 2 in [4].

Proposition. *Let H be a compact quantum group with respect to * and let $x \mapsto x^\#$ be another structure of *-Hopf algebra on H. Then there exists a Hopf algebra automorphism T of H such that $\#$ and $T * T^{-1}$ commute.*

Proof. Let N be given by $N(u) = (u^*)^\#$; this is a Hopf algebra automorphism and any finite dimensional submodule of H is contained in some finite dimensional submodule W such that $N(W) = W$. By Proposition 2.4, the Hermitian form $(\,|\,)_\ell$ (defined with respect to *) is positive definite. From Lemma 1.7, we deduce that N is self-adjoint with respect to $(\,|\,)_\ell$. Then the Hopf algebra automorphism $P = N^2$ is diagonalizable with positive eigenvalues; let $(X_i)_{i\in I}$ be a basis of H such that $PX_i = \lambda_i X_i$. For each $s \in \mathbb{R}$, one has a well-defined linear automorphism P^s of H. We claim that P^s is also a Hopf algebra automorphism. Let c_{ij}^k be constants such that $\Delta(X_k) = \sum_{i,j} c_{ij}^k X_i \otimes X_j$, for all k. Hence

$$\lambda_i \lambda_j c_{ij}^k = \lambda_k c_{ij}^k$$

for all i, j, k and *a fortiori* $\lambda_i^s \lambda_j^s c_{ij}^k = \lambda_k^s c_{ij}^k$, that is, P^s preserves the comultiplication. With similar arguments, one shows that P^s is a morphism of Hopf algebras. Now $T = P^{1/4}$ does the job, *cf.* p. 183 in [6]. □

Theorem 2.6. *Let H be a compact quantum group with respect to * and also with respect to $\#$. Then there exists a Hopf algebra automorphism T such such that $*T = T\#$.*

Proof. Taking into account that H_ρ is *- and $\#$-stable, the proof in [6], p. 184, (see also [4]) can be adapted here. □

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